JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **34**, No. 1, February 2021
http://dx.doi.org/10.14403/jcms.2021.34.1.17

ON THE CRYSTALLOGRAPHIC GROUP OF $\mathrm{Sol}^4_{m,n}$

Won Sok Yoo

Abstract. The purpose of this paper is to determine the structure of the crystallographic group Π of the 4-dimensional solvable Lie group $\mathrm{Sol}^4_{m,n}$ that the translation subgroup of Π, $\Gamma := \Pi \cap \mathrm{Sol}^4_{m,n}$, is generated by the particular elements.

1. Introduction

Let X be a complete connected, simply connected Riemannian manifold, and let G be a group of isometries of X. A pair (X, G) is called a *geometry* in the sense of Thurston [6, 7] if G acts transitively on X and G contains a discrete subgroup Γ with the coset space $\Gamma\backslash X$ of finite volume. According to Filipkiewicz [3, 9], there are 20 types of geometries in dimension 4: $S^4, \mathbb{H}^4, P^2(\mathbb{C}), H^2(\mathbb{C}), S^2\times S^2, S^2\times\mathbb{R}^2, S^2\times \mathbb{H}^2, \mathbb{R}^4, \mathbb{R}^2\times\mathbb{H}^2, \mathbb{H}^2\times\mathbb{H}^2, S^3\times\mathbb{R}, \mathbb{H}^3\times\mathbb{R}, \widetilde{\mathrm{PSL}}(2,\mathbb{R})\times\mathbb{R}, \mathrm{Nil}^3\times\mathbb{R}, \mathrm{Sol}^3\times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}^4_{m,n}, \mathrm{Sol}^4_0, \mathrm{Sol}^4_1$ and F^4.

Let G be a connected, simply connected solvable Lie group and let C be any maximal compact subgroup of $\mathrm{Aff}(G)$. A discrete cocompact subgroup Π of $G \rtimes C$ is called a *crystallographic group* of G. The coset space $\Pi\backslash G$ is an *infra-solvmanifold* of G, when Π is a *Bieberbach group* (i.e., a torsion-free crystallographic group) of G. The maximal compact subgroup C can be chosen so that $G \rtimes C$ is equal to $\mathrm{Isom}(G)$. Therefore, the Bieberbach groups of G are exactly the fundamental groups of compact infra-solvmanifolds of G. Consequently, a closed manifold has a (X, G)-geometry if and only if it is an infra-solvmanifold of G. The crystallographic groups of Sol^3 and Sol^4_1 are classified in [2] and [4], respectively. All the closed four-manifolds with Sol^4_1-geometry were

Received December 07, 2020; Accepted December 28, 2020.
2010 Mathematics Subject Classification: 57S25, 22E25.
Key words and phrases: Bieberbach group, crystallographic group, infra-solvmanifold, $\mathrm{Sol}^4_{m,n}$.
This research was supported by Kumoh National Institute of Technology (202001960001).

studied in [8]. Utilizing the ideas in [2, 4, 8], the aim of this paper is to determine the structure of the crystallographic group of the solvable Lie group $\mathrm{Sol}^4_{m,n}$.

This paper is organized as follows. In Section 2, we show that a compact subgroup of the group of automorphisms of the Lie group Sol^4_λ has at most 8 elements. There are an infinite but countable number of the Lie groups Sol^4_λ that admit a lattice. Such Lie groups are denoted by $\mathrm{Sol}^4_{m,n}$. In Section 3, we review a family of Lie groups $\mathrm{Sol}^4_{m,n}$. In Section 4, we study the structure of the crystallographic group Π of $\mathrm{Sol}^4_{m,n}$ that the translation subgroup of Π, $\Gamma := \Pi \cap \mathrm{Sol}^4_{m,n}$, is generated by the particular elements.

2. The Lie group Sol^4_λ and its automorphism group

The Lie group Sol^4_λ is a 4-dimensional connected, simply connected and unimodular solvable Lie group $\mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ of type (R) where

$$\varphi(s) = \begin{bmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-(1+\lambda)s} \end{bmatrix} \quad (\lambda > 1).$$

This can be embedded in $\mathrm{Aff}(4)$ as

$$\mathrm{Sol}^4_\lambda = \left\{ \begin{bmatrix} \varphi(s) & 0 & \mathbf{x} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset \mathrm{Aff}(4) \subset \mathrm{GL}(5, \mathbb{R}),$$

where $\mathbf{x} \in \mathbb{R}^3$ is a column vector. The Lie algebra $\mathfrak{sol}^4_\lambda$ of Sol^4_λ is

$$\mathfrak{sol}^4_\lambda = \left\{ \begin{bmatrix} \tau(s) & 0 & \mathbf{a} \\ 0 & 0 & s \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

where

$$\tau(s) = \log \varphi(s) = \begin{bmatrix} \lambda s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -(1+\lambda)s \end{bmatrix}.$$

Now let us first find the group of automorphisms $\mathrm{Aut}(\mathrm{Sol}^4_\lambda)$ of Sol^4_λ. Because Sol^4_λ is simply connected, it suffices to find the group of Lie algebra automorphisms of the Lie algebra $\mathfrak{sol}^4_\lambda$. For this purpose, we choose a linear basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{F}\}$ of $\mathfrak{sol}^4_\lambda$ as follows:

$$\mathbf{E}_i = \begin{bmatrix} \tau(0) & 0 & \mathbf{e}_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} \tau(1) & 0 & \mathbf{0} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the set of standard basis vectors of $\mathbb{R}^3$. Then the nontrivial Lie brackets between them are

$$[\mathbf{F}, \mathbf{E}_1] = \lambda \mathbf{E}_1, \ [\mathbf{F}, \mathbf{E}_2] = \mathbf{E}_2, \ [\mathbf{F}, \mathbf{E}_3] = -(1+\lambda)\mathbf{E}_3. \tag{2–1}$$

A Lie algebra automorphism of $\mathfrak{sol}^4_\lambda$ is a nonsingular linear transformation of the linear space $\mathfrak{sol}^4_\lambda$ preserving the nontrivial Lie brackets (2–1) together with all trivial Lie brackets. It is now easy to observe that:

Proposition 2.1. *The Lie group* $\mathrm{Aut}(\mathfrak{sol}^4_\lambda)$ *is, with respect to the linear basis* $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{F}\}$, *the following matrix group*

$$\left\{ \begin{bmatrix} a & 0 & 0 & * \\ 0 & b & 0 & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \,\middle|\, abc \neq 0 \right\} \cong \mathbb{R}^3 \rtimes \mathrm{GD}(3),$$

where $\mathrm{GD}(3)$ *is the group of all invertible* 3×3*-diagonal matrices and it acts on* $\mathbb{R}^3$ *by matrix multiplication.*

From Proposition 2.1, it is immediate that a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{sol}^4_\lambda)$ is

$$\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1) = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\} \cong (\mathbb{Z}_2)^3$$

which is a maximal compact subgroup of $\mathrm{GD}(3)$.

Remark that the Lie group Sol^4_λ is of type (R) and hence is of type (E), that is, the exponential map $\exp : \mathrm{Sol}^4_\lambda \to \mathfrak{sol}^4_\lambda$ is a diffeomorphism. Using this diffeomorphism, we can observe that $\begin{bmatrix} a & 0 & 0 & p \\ 0 & b & 0 & q \\ 0 & 0 & c & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{sol}^4_\lambda)$

is an automorphism of Sol_λ^4 given by

$$\begin{bmatrix} e^{\lambda s} & 0 & 0 & 0 & x \\ 0 & e^s & 0 & 0 & y \\ 0 & 0 & e^{-(1+\lambda)s} & 0 & z \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} e^{\lambda s} & 0 & 0 & 0 & ax + sp\frac{e^{\lambda s}-1}{\lambda s} \\ 0 & e^s & 0 & 0 & by + sq\frac{e^{s}-1}{s} \\ 0 & 0 & e^{-(1+\lambda)s} & 0 & cz + sr\frac{e^{-(1+\lambda)s}-1}{-(1+\lambda)s} \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, $\mathrm{GD}(3)$ ($p = q = r = 0$) acts on $\mathrm{Sol}_\lambda^4 = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$ as matrix multiplication on its nilradical $\mathbb{R}^3$. Consequently,

$$\text{(2–2)} \qquad \mathrm{Sol}_\lambda^4 \rtimes \mathrm{GD}(3) = (\mathbb{R}^3 \rtimes_\varphi \mathbb{R}) \rtimes \mathrm{GD}(3) = \mathbb{R}^3 \rtimes_{\varphi'} (\mathbb{R} \times \mathrm{GD}(3))$$

where $\varphi'(s, X) = \varphi(s) \cdot X = X \cdot \varphi(s)$.

3. The Lie group $\mathrm{Sol}_{m,n}^4$

In this section, we will briefly review a family of Lie groups $\mathrm{Sol}_{m,n}^4$. A good reference is [5] or [9]. Let Γ be a lattice (i.e., a discrete cocompact subgroup) of $\mathrm{Sol}_\lambda^4 = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$. Then $\Gamma \cap \mathbb{R}^3$ is a lattice of $\mathbb{R}^3$ and $\Gamma/(\Gamma \cap \mathbb{R}^3)$ is a lattice of $\mathrm{Sol}_\lambda^4/\mathbb{R}^3 = \mathbb{R}$, so that $\Gamma \cap \mathbb{R}^3 \cong \mathbb{Z}^3$ and $\Gamma/(\Gamma \cap \mathbb{R}^3) \cong \mathbb{Z}$, and the following diagram of short exact sequences is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \rtimes_\varphi \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

The rightmost map is injective. We may assume this injective map is an inclusion $\mathbb{Z} \subset \mathbb{R}$. Choose a generator $s > 0$ of the group $\mathbb{Z}$. Then $\mathbb{Z}^3$ is a $\varphi(s)$-invariant lattice of $\mathbb{R}^3$, namely, $\varphi(s)$ can be regarded as an automorphism on $\mathbb{Z}^3$. Choose a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ of $\mathbb{Z}^3$. Then we must have that

$$\text{(3–1)} \qquad \varphi(s)(\mathbf{x}_i) = \ell_{1i}\mathbf{x}_1 + \ell_{2i}\mathbf{x}_2 + \ell_{3i}\mathbf{x}_3, \ (i = 1, 2, 3)$$

for some integers ℓ_{ij}. Thus the lattice Γ is a subgroup of Sol_λ^4 generated by the following elements

$$\mathbf{x}_1 = (\mathbf{x}_1, 0), \ \mathbf{x}_2 = (\mathbf{x}_2, 0), \ \mathbf{x}_3 = (\mathbf{x}_3, 0), \ s = (\mathbf{0}, s)$$

of $\mathrm{Sol}_\lambda^4 = \mathbb{R}^3 \rtimes_\varphi \mathbb{R}$. We shall denote such a lattice by

(3–2)
$$\Gamma = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, s \mid [\mathbf{x}_i, \mathbf{x}_j] = 1, \ \varphi(s)(\mathbf{x}_i) = \ell_{1i}\mathbf{x}_1 + \ell_{2i}\mathbf{x}_2 + \ell_{3i}\mathbf{x}_3 \rangle.$$

Let

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}.$$

Then $\Gamma \cong \mathbb{Z}^3 \rtimes_A \mathbb{Z}$.

Now we form the matrix P with columns $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$. Then (3–1) is equivalent to

$$PAP^{-1} = \varphi(s) = \begin{bmatrix} e^{\lambda s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-(1+\lambda)s} \end{bmatrix}. \tag{3–3}$$

This implies that $A \in \mathrm{SL}(3, \mathbb{Z})$ and the columns of P^{-1} are eigenvectors of A with corresponding eigenvalues $e^{\lambda s}, e^s$ and $e^{-(1+\lambda)s}$, respectively.

For another basis $\{\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3\}$ of $\mathbb{Z}^3$, we let P' be the matrix with columns $\mathbf{x}'_1, \mathbf{x}'_2$ and $\mathbf{x}'_3$. Then we have that

$$\varphi(s)(\mathbf{x}'_i) = \ell'_{1i}\mathbf{x}'_1 + \ell'_{2i}\mathbf{x}'_2 + \ell'_{3i}\mathbf{x}'_3, \ (i = 1, 2, 3)$$

for some integers ℓ'_{ij}. If

$$A' = \begin{bmatrix} \ell'_{11} & \ell'_{12} & \ell'_{13} \\ \ell'_{21} & \ell'_{22} & \ell'_{23} \\ \ell'_{31} & \ell'_{32} & \ell'_{33} \end{bmatrix},$$

then we have

$$\varphi(s) = P'A'P'^{-1}.$$

Therefore, $A' = P'^{-1}PAP^{-1}P'$, i.e., A and A' are conjugate by an element of $\mathrm{GL}(3, \mathbb{R})$. Clearly, $\mathbb{Z}^3 \rtimes_A \mathbb{Z} \cong \mathbb{Z}^3 \rtimes_{A'} \mathbb{Z}$.

Let

$$\chi_A(x) = x^3 - mx^2 + nx - 1$$

be the characteristic polynomial of A (so $m, n \in \mathbb{Z}$). Since A and $\varphi(s)$ are conjugate, we have

$$\begin{aligned} m &= e^{\lambda s} + e^s + e^{-(1+\lambda)s} = \mathrm{tr}(A) \\ n &= e^{-\lambda s} + e^{-s} + e^{(1+\lambda)s} = \mathrm{tr}(A^{-1}). \end{aligned}$$

Note that $m > 3$. [It can be seen that the function $f(x) = e^{\lambda x} + e^x + e^{-(1+\lambda)x}$ has the global minimum value 3 at $x = 0$.] Similarly, $n > 3$. We call such Sol^4_λ as $\mathrm{Sol}^4_{m,n}$.

By choosing $-s$ as another generator of the group $\mathbb{Z}(\subset \mathbb{R})$, we see that $\mathrm{Sol}^4_{n,m} \cong \mathrm{Sol}^4_{m,n}$. Note also that e^s cannot be 1, that is, 1 cannot be a root of $\chi_A(x)$, which happens when and only when $m = n$. Remark that $\mathrm{Sol}^4_{m,m} \cong \mathrm{Sol}^3 \times \mathbb{R}$. Thus in what follows we shall assume that $m > n > 3$.

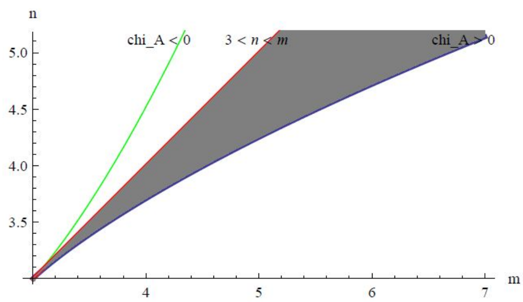
Since $m > n > 3$, we have $m^2 - 3n > 0$. So, the characteristic polynomial $\chi_A(x)$ of A has two positive critical values

$$x = \frac{1}{3}\left(m \pm \sqrt{m^2 - 3n}\right).$$

Then $\chi_A(x)$ has 3 distinct positive real roots if and only if

$$\begin{aligned} &m > n > 3, \\ &\chi_A\left(\tfrac{1}{3}\left(m - \sqrt{m^2 - 3n}\right)\right) > 0, \\ &\chi_A\left(\tfrac{1}{3}\left(m + \sqrt{m^2 - 3n}\right)\right) < 0. \end{aligned} \tag{3–4}$$

Consequently, if the group Sol^4_χ has a lattice, then there exists a pair of integers (m, n) satisfying the conditions (3–4), or equivalently, lying in the shaded region.



Conversely, suppose (m, n) is a pair of integers satisfying the conditions (3–4). Then the equation $x^3 - mx^2 + nx - 1 = 0$ has 3 distinct positive real roots, say $\alpha_1 > \alpha_2 > \alpha_3 > 0$. This equation has the companion matrix

$$A_{m,n} := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{bmatrix}.$$

Let P be the Vandermonde matrix corresponding $\alpha_1, \alpha_2, \alpha_3$:

$$P = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{bmatrix}.$$

Then

$$P\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & -n\\ 0 & 1 & m\end{bmatrix} = \begin{bmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{bmatrix} P.$$

Moreover,

$$\begin{aligned}&\alpha_1\alpha_2\alpha_3 = 1\\ &\alpha_1 + \alpha_2 + \alpha_3 = m\\ &\alpha_1^{-1} + \alpha_2^{-1} + \alpha_2^{-1} = \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 = n\end{aligned}$$

Consequently, if $A \in \mathrm{SL}(3,\mathbb{Z})$ has characteristic polynomial $\chi_A(x) = x^3 - mx^2 + nx - 1$ then it is conjugate to $A_{m,n}$.

Let

$$\lambda = \frac{\ln\alpha_1}{\ln\alpha_2},\ \ s = \ln\alpha_2.$$

A direct computation shows that $Q = P^{-1}$ is, up to a nonzero constant,

$$\begin{bmatrix}-\alpha_2\alpha_3(\alpha_2-\alpha_3) & -\alpha_3\alpha_1(\alpha_3-\alpha_1) & -\alpha_1\alpha_2(\alpha_1-\alpha_2)\\ (\alpha_2+\alpha_3)(\alpha_2-\alpha_3) & (\alpha_3+\alpha_1)(\alpha_3-\alpha_1) & (\alpha_1+\alpha_2)(\alpha_1-\alpha_2)\\ -(\alpha_2-\alpha_3) & -(\alpha_3-\alpha_1) & -(\alpha_1-\alpha_2)\end{bmatrix}.$$

Hence the vectors

$$\text{(3–5)}\quad \mathbf{x}_1 = \begin{bmatrix}\alpha_2\alpha_3\\ -(\alpha_2+\alpha_3)\\ 1\end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix}\alpha_3\alpha_1\\ -(\alpha_3+\alpha_1)\\ 1\end{bmatrix},\ \mathbf{x}_3 = \begin{bmatrix}\alpha_1\alpha_2\\ -(\alpha_1+\alpha_2)\\ 1\end{bmatrix}$$

are eigenvectors of $A_{m,n}$ with eigenvalues $\alpha_1 = e^{\lambda s}, \alpha_2 = e^s$ and $\alpha_3 = e^{-(1+\lambda)s}$, respectively. This proves that the abstract group $\mathbb{Z}^3 \rtimes_{A_{m,n}} \mathbb{Z}$ is isomorphic to the lattice of $\mathrm{Sol}^4_\lambda = \mathrm{Sol}^4_{m,n}$ generated by $(\mathbf{x}_1, 0), (\mathbf{x}_2, 0), (\mathbf{x}_3, 0)$ and $(\mathbf{0}, s)$, see (3–2).

4. The structure of crystallographic group of $\mathrm{Sol}^4_{m,n}$

We recall that a closed 4-dimensional manifold M has $\mathrm{Sol}^4_{m,n}$-*geometry* if and only if it is an infra-solvmanifold of $\mathrm{Sol}^4_{m,n}$, $M = \Pi\backslash\mathrm{Sol}^4_{m,n}$. Therefore, Π is a torsion-free discrete cocompact subgroup of $\mathrm{Sol}^4_{m,n} \rtimes K \subset \mathrm{Aff}(\mathrm{Sol}^4_{m,n})$ where $K = \mathrm{O}(1)^3$ is a maximal compact subgroup of the group of automorphisms $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ of $\mathrm{Sol}^4_{m,n}$.

Let Π be a crystallographic group of $\mathrm{Sol}^4_{m,n}$. Since the Bieberbach theorems generalize to $\mathrm{Sol}^4_{m,n}$ [1], the translation subgroup of Π, $\Gamma := \Pi\cap\mathrm{Sol}^4_{m,n}$, is of finite index in Π, and is a lattice of $\mathrm{Sol}^4_{m,n}$. The maximal compact subgroup K is very small, has only 8 elements. Therefore, all crystallographic groups of $\mathrm{Sol}^4_{m,n}$ are extensions of a lattice by a subgroup Φ of the finite group K.

Given a pair of integers (m,n) satisfying the conditions (3–4), we denote by $\alpha_1 > \alpha_2 > \alpha_3 > 0$ the 3 distinct positive real roots of the associated equation $x^3 - mx^2 + nx - 1 = 0$.

The purpose of this paper is to determine the structure of the crystallographic group Π of which the translation subgroup Γ is generated by $\{(\mathbf{x}_1, 0), (\mathbf{x}_2, 0), (\mathbf{x}_3, 0), (\mathbf{0}, s)\}$, where $s = \ln \alpha_2$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are given as in (3–5). With $\Phi := \Pi/\Gamma \subset K$ and $\mathbb{Z}_\Phi := \Pi/(\Gamma \cap \mathbb{R}^3)$, we obtain the commutative diagram below with exact rows and columns

(4–1)
$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^3 & \xrightarrow{=} & \mathbb{Z}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

where $\mathbb{Z}^3 = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ and $\mathbb{Z} = \langle s \rangle$.

Remark also that $\Pi \subset \mathrm{Sol}^4_{m,n} \rtimes K = \mathbb{R}^3 \rtimes_{\varphi'} (\mathbb{R} \times K)$ and hence $\mathbb{Z}_\Phi \subset \mathbb{R} \times K$. In particular, $\mathbb{Z}_\Phi$ is abelian. Consequently, it makes easy to determine $\mathbb{Z}_\Phi \subset \mathbb{R} \times K$, which is an extension of $\mathbb{Z} = \langle s \rangle$ by Φ.

For any non-trivial element X of Φ, there exists a $t \in \mathbb{R}$ such that $(t, X) \in \mathbb{Z}_\Phi$. Since $(t, X)^2 = (2t, X^2) = (2t, I) \in \langle s \rangle$, we may assume that $t = 0$ or $t = \frac{s}{2}$. Hence the subgroup $\langle (s, I), (t, X) \rangle$ is either $\langle (s, I), (0, X) \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$ or $\langle (\frac{s}{2}, X) \rangle \cong \mathbb{Z}$.

Let $X, Y \in \Phi$ generate a subgroup of Φ isomorphic to $\mathbb{Z}_2^2$. Choose lifts $(t, X), (u, Y) \in \mathbb{Z}_\Phi$ of X, Y where t, u are 0 or $\frac{s}{2}$. Therefore, the subgroup $\langle (s,I),(t,X),(u,Y)\rangle$ of $\mathbb{Z}_\Phi$ is one of the following:

$$\begin{aligned}
&\langle (s,I),(0,X),(0,Y)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (\tfrac{s}{2},X),(0,Y)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2,\\
&\langle (0,X),(\tfrac{s}{2},Y)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2,\\
&\langle (\tfrac{s}{2},X),(\tfrac{s}{2},Y)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2.
\end{aligned}$$

Let $\Phi = K$ with generators X, Y, Z. Choose lifts $(t,X),(u,Y),(v,Z)\in \mathbb{Z}_\Phi$ of X,Y,Z where t,u and v are 0 or $\frac{s}{2}$. Then it can be seen easily that the subgroup $\langle (s,I),(t,X),(u,Y),(v,Z)\rangle$ of $\mathbb{Z}_\Phi$ is one of the following:

$$\begin{aligned}
&\langle (s,I),(0,X),(0,Y),(0,Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^3,\\
&\langle (\tfrac{s}{2},X),(0,Y),(0,Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (0,X),(\tfrac{s}{2},Y),(0,Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (0,X),(0,Y),(\tfrac{s}{2},Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (0,X),(\tfrac{s}{2},Y),(\tfrac{s}{2},Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (\tfrac{s}{2},X),(0,Y),(\tfrac{s}{2},Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (\tfrac{s}{2},X),(\tfrac{s}{2},Y),(0,Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2,\\
&\langle (\tfrac{s}{2},X),(\tfrac{s}{2},Y),(\tfrac{s}{2},Z)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2^2.
\end{aligned}$$

In conclusion, we can see that $\mathbb{Z}_\Phi$ is isomorphic to one of the following

$$\mathbb{Z},\ \mathbb{Z}\times\mathbb{Z}_2,\ \mathbb{Z}\times\mathbb{Z}_2^2,\ \mathbb{Z}\times\mathbb{Z}_2^3.$$

Moreover, if Φ has 2 or more generators, then $(0,X)\in\mathbb{Z}_\Phi$ for some nontrivial element X of Φ.

We know all the subgroups Φ of $K\subset \mathrm{Aut}(\mathrm{Sol}^4_{m,n})$. The group $K = \mathrm{O}(1)^3$ is generated by

$$X=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ Y=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\ Z=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}.$$

Thus the nontrivial subgroups Φ of $K=\{\pm I,\pm X,\pm Y,\pm Z\}$ are:

Φ	generator(s)
$\mathbb{Z}_2$	$\langle X\rangle,\ \langle Y\rangle,\ \langle Z\rangle,\ \langle -X\rangle,\ \langle -Y\rangle,\ \langle -Z\rangle,\ \langle -I\rangle$
$\mathbb{Z}_2^2$	$\langle X,-X\rangle,\ \langle X,Y\rangle,\ \langle X,Z\rangle,\ \langle Y,-Y\rangle,\ \langle Y,Z\rangle,\ \langle Z,-Z\rangle,\ \langle -X,-Z\rangle$
$\mathbb{Z}_2^3$	$\langle X,Y,Z\rangle$

References

[1] K. Dekimpe, K. B. Lee and F. Raymond, *Bieberbach theorems for solvable Lie groups*, Asian J. Math., **5** (2001), 499-508.

[2] K. Y. Ha and J. B. Lee, *Crystallographic groups of* Sol, Math. Nachr., **286** (2013), 1614-1667.

[3] J. A. Hillman, *Four-manifolds , Geometries and Knots*, Geometry & Topology Monographs, 5, Geometry & Topology Publications, Coventry, 2002.

[4] K. B. Lee and S. Thuong, *Infra-solvmanifolds of* Sol_1^4, J. Korean Math. Soc., **52** (2015), 1209-1251.

[5] J. B. Lee, K. B. Lee, J. Shin and S. Yi, *Unimodular groups of type* $\mathbb{R}^3 \rtimes \mathbb{R}$, J. Korean Math. Soc., **44** (2007), 1121-1137.

[6] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc., **15** (1983), 401-487.

[7] W. P. Thurston, *Three-dimensional Geometry and Topology*, **1**, Princeton University Press, Princeton, NJ, 1997.

[8] S. V. Thuong, *Classification of closed manifolds with* Sol_1^4*-geometry*, Geom. Dedicata., **199** (2019), 373-397.

[9] C. T. C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology, **25** (1986), 119-153.

Department of Applied Mathematics
Kumoh National Institute of Technology
Kumi 39177, Republic of Korea
E-mail: wsyoo@kumoh.ac.kr